On Generalized Fibonacci Quaternions and Fibonacci-Narayana Quaternions

Cristina FLAUT and Vitalii SHPAKIVSKYI

Abstract. In this paper, we investigate some properties of generalized Fibonacci quaternions and Fibonacci-Narayana quaternions.

Keywords: Fibonacci quaternions, generalized Fibonacci quaternions, Fibonacci-Narayana quaternions.
2000 AMS Subject Classification: 11B83, 11B99.

0. Introduction

The Fibonacci numbers was introduced by *Leonardo of Pisa (1170-1250)* in his book *Liber abbaci*, book published in 1202 AD. These numbers was used as a model for investigate the growth of rabbit populations. (see [Dr, Gi, Gr, Wa; 03]) The latin name of Leonardo was *Leonardus Pisanus*, also called *Leonardus filius Bonaccii*, shortly *Fibonacci*. This name is attached to the following sequence of numbers

$$0, 1, 1, 2, 3, 5, 8, 13, 21, ...,$$

with the nth term given by the formula:

$$f_n = f_{n-1} + f_{n-2,}\ n \geq 2,$$

where $f_0 = 0, f_1 = 1$.

Fibonacci numbers was known in India before Leonardo's time and used by the Indian authorities on metrical sciences. (see [Pa; 85], p. 230). These numbers have many properties which was studied by many authors. (see [Ho; 61], [Cu; 76], [Pa; 85], [Ko; 01])

Narayana was an outstanding Indian mathematician of the XIV century. From him came to us the manuscript "Bidzhahanity" (incomplete), written in the middle of the XIV century. For Narayana was interesting summation of arithmetic series and magic squares. In the middle of of the XIV century he proved a more general summation. Using the following sums

$$1 + 2 + 3 + \ldots + n = S_n^{(1)},$$
$$S_1^{(1)} + S_2^{(1)} + \ldots + S_n^{(1)} = S_n^{(2)},$$
$$S_1^{(2)} + S_2^{(2)} + \ldots + S_n^{(2)} = S_n^{(2)}, \ldots ,$$

Narayana calculated that

$$S_n^{(m)} = \frac{n(n+1)(n+2)\ldots(n+m)}{1 \cdot 2 \cdot 3 \cdot \ldots \cdot (m+1)} \;. \qquad (*)$$

Narayana applied its rules to the problem of a herd of cows and heifers. (see [Yu; 61])

Narayana problem. *A cow annually brings heifers. Every heifer, beginning from the fourth year of his life also brings heifer . How many cows and calves will be after 20 years?*

Narayana's calculation is in the following:

1) a cow within 20 years brings 20 heifers of the first generation;

2) the first heifer of the first generation brings 17 heifers second generation, the second heifer of the first generation brings 16 heifers second generation etc. The total in the second generation will be $17 + 16 + \ldots + 1 = S_{17}^{(1)}$ cows and calves;

3) the first heifer of the seventeen heifers of the second generation brings 14 heifers of the third generation, the second heifer of the seventeen heifers of the second generation brings 13 heifers of third generation, etc. The total heifers of the first generation brings $13 + 12 + \ldots + 1 = S_{13}^{(1)}$ heads. Now, all heifers of the second generation brings $S_{14}^{(1)} + S_{13}^{(1)} + \ldots + S_1^{(1)} = S_{14}^{(2)}$ heads in the third generation.

Similarly, Narayana calculated total number in the herd after 20 years:

$$n = 1 + 20 + S_{17}^{(1)} + S_{14}^{(2)} + \ldots + S_{2}^{(6)}.$$

Using formula (*), he obtained:

$$n = 1 + 20 + \frac{17 \cdot 18}{1 \cdot 2} + \frac{14 \cdot 15 \cdot 16}{1 \cdot 2 \cdot 3} + \frac{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} = 2745.$$

This problem can be solved in the same way that Fibonacci solved its problem about rabbits (see [Ko; 01]):

In the beginning of the first year was 1 cow and 1 heifer which born. That is had 2 heads. In the beginning of the second year and in the beginning of the third year the number of heads increased by one. Therefore the number of heads are 3 and 4, respectively. From the fourth year, the number of heads in the herd is defined by recurrence formulae:

$$x_4 = x_3 + x_1, x_5 = x_4 + x_2, \ldots, x_n = x_{n-1} + x_{n-3},$$

since the number of cows for any year is equal with the number of cows of the previous year plus the number of heifers which was born (= number of heads that were three years ago).

We have the sequence

$$2, 3, 4, 6, 9, \ldots, u_{n+1} = u_n + u_{n-2}.$$

Computing, we obtain that $u_{20} = 2745$.

We can consider now the sequence

$$1, 1, 1, 2, 3, 4, 6, 9, \ldots, u_{n+1} = u_n + u_{n-2},$$

which is called the *Fibonacci-Narayana sequence*. (see [Di, St; 03])

In the same paper [Di, St; 03], authors proved some basic properties of Fibonacci-Narayana numbers, namely:

1) $u_1 + u_2 + \ldots + u_n = u_{n+3} - 1$.
2) $u_1 + u_4 + u_7 + \ldots + u_{3n-2} = u_{3n-1}$.
3) $u_2 + u_5 + u_8 + \ldots + u_{3n-1} = u_{3n}$.
4) $u_3 + u_6 + u_9 + \ldots + u_{3n} = u_{3n+1} - 1$.
5) $u_{n+m} = u_{n-1}u_{m+2} + u_{n-2}u_m + u_{n-3}u_{m+1}$.
6) $u_{2n} = u_{n+1}^2 + u_{n-1}^2 + u_{n-2}^2$.

7) If in the sequences $\{u_n\}$, $n = 7k+4$, $n = 7k+6$, $n = 7k$, when $k = 0, 1, 2, \ldots$, then u_n is even.

8) If in the sequences $\{u_n\}$ $n = 8k$, $n = 8k-1$, $n = 8k-3$, when $k = 0, 1, 2, \ldots$, then $u_n \,\vdots\, 3$,

Another propertiy of Fibonacci-Narayana numbers was proved in [Sh; 06]: for all natural $n \geq 2$ we have

$$u_n = \sum_{m=0}^{[n/3]} \mathrm{C}_{[n/3]}^{m} u_{n-[n/3]-2m},$$

where $[a]$ is an integer part of a.

Let $\mathbb{H}$ be the real division quaternion algebra, the algebra of the elements of the form $a = a_1 \cdot 1 + a_2 e_2 + a_3 e_3 + a_4 e_4$, where

$$a_i \in \mathbb{R}, i \in \{1,2,3,4\}, e_i^2 = -1, \;\; i \in \{2,3,4\}$$

and

$$e_i e_j = -e_j e_i = \beta_{ij} e_k, \;\; \beta_{ij} \in \{-1, 1\}, i \neq j, i, j \in \{\, 2, 3, 4\}.$$

β_{ij} and e_k being uniquely determined by e_i and e_j.

$\mathbb{H}$ is an algebra over the field $\mathbb{R}$ and the set $\{1, e_2, e_3, e_4\}$ is a basis in $\mathbb{H}$. The conjugate of the real quaternion $a = a_1 \cdot 1 + a_2 e_2 + a_3 e_3 + a_4 e_4$, is the quaternion $\overline{a} = a_1 \cdot 1 - a_2 e_2 - a_3 e_3 - a_4 e_4$, $n(a) = a\overline{a} = \overline{a}a$ is called *the norm* of the real quaternion a and $t(a) = 2a_1$ is called *the trace* of the real quaternion a.The relation $a^2 - t(a)\,a + n(a) = 0$ is fulfilled for all real quaternions.

In [Ho; 63], Horadam defined the Fibonacci quaternions as

$$F_n = f_n \cdot 1 + f_{n+1} e_2 + f_{n+2} e_3 + f_{n+3} e_4,$$

where f_n are the n-th Fibonacci number and found a lot of properties of them. After that, many authors studied Fibonacci and generalized Fibonacci quaternions giving more and surprising new properties. (for example, see [Sw; 73], [Sa-Mu; 82] and [Ha; 12])

Similar to A. F. Horadam, we define the Fibonacci-Narayana quaternions as

$$U_n = u_n \cdot 1 + u_{n+1} e_2 + u_{n+2} e_3 + u_{n+3} e_4,$$

where u_n are the n-th Fibonacci-Narayana number.

In this paper, we give some properties of generalized Fibonacci quaternions and Fibonacci-Narayana quaternions.

1. Preliminaries

In the present days, several mathematicians studied the properties of the Fibonacci sequence. In [Ho; 61], the author generalized the Fibonacci numbers and gave many properties of them:

$$h_n = h_{n-1} + h_{n-2}, \quad n \geq 2,$$

where $h_0 = p, h_1 = q$, with p, q being arbitrary integers. The same author, in [Ho; 63], defined and studied Fibonacci quaternions and generalized Fibonacci quaternions, given by the formulae:

$$F_n = f_n \cdot 1 + f_{n+1}e_2 + f_{n+2}e_3 + f_{n+3}e_4,$$

for the nth Fibonacci quaternions, and

$$H_n = h_n \cdot 1 + h_{n+1}e_2 + h_{n+2}e_3 + h_{n+3}e_4,$$

for the nth generalized Fibonacci quaternions, where

$$e_i^2 = -1, \;\; i \in \{2, 3, 4\}$$

and

$$e_i e_j = -e_j e_i = \beta_{ij} e_k, \;\; \beta_{ij} \in \{-1, 1\}, i \neq j, i, j \in \{\, 2, 3, 4\}.$$

β_{ij} and e_k being uniquely determined by e_i and e_j. In the same paper, the author gave a few relations, as for example the norm formula for the nth Fibonacci quaternions:

$$n\left(F_n\right) = F_n \overline{F}_n = 3F_{2n+3},$$

where $\overline{F}_n = f_n \cdot 1 - f_{n+1}e_2 - f_{n+2}e_3 - f_{n+3}e_4$ is the conjugate of the F_n.

We remark that

$$h_{n+1} = pf_n + qf_{n+1}.$$

Similar to A. F. Horadam, we define the Fibonacci-Narayana quaternions as

$$U_n = u_n \cdot 1 + u_{n+1}e_2 + u_{n+2}e_3 + u_{n+3}e_4,$$

where u_n are the n-th Fibonacci-Narayana number.

M. N. S. Swamy, in [Sw; 73], obtained some relations for the nth generalized Fibonacci quaternions. One of them, which is important in the following, is refers to the norm formula:

$$\begin{aligned} n(H_n) &= H_n\overline{H}_n = \\ &= 3\left(p^2 f_{2n+3} + 2pq f_{2n+2} + q^2 f_{2n+1}\right) = \\ &= 3[(p^2+2pq)f_{2n+2} + (p^2+q^2)f_{2n+1}] \end{aligned}$$

where $\overline{H}_n = h_n \cdot 1 - h_{n+1}e_2 - h_{n+2}e_3 - h_{n+3}e_4$ is the conjugate of the H_n.

2. Generalized Fibonacci Quaternions

Let $F_n = f_n \cdot 1 + f_{n+1}e_2 + f_{n+2}e_3 + f_{n+3}e_4$ be the nth Fibonacci quaternion.

Theorem 2.1. *For the Fibonacci quaternion F_n*, we have

$$\sum_{m=0}^{n} (-1)^{m+1} F_m = (-1)^n [F_{n-1} + 1 + e_2 + e_3 + e_4] \tag{2.1}$$

Proof.

It results:

$\sum\limits_{m=0}^{n} (-1)^{m+1} F_m =$

$= \sum\limits_{m=0}^{n} (-1)^{m+1} f_m + e_2 \sum\limits_{m=0}^{n} (-1)^{m+1} f_{m+1} +$

$+ e_3 \sum\limits_{m=0}^{n} (-1)^{m+1} f_{m+2} + e_4 \sum\limits_{m=0}^{n} (-1)^{m+1} f_{m+3} =$

$= (-1)^n (f_{n-1}+1) - (-1)^{n+1} (f_n+1) e_2 +$

$+ (-1)^{n+2} (f_{n+1}+1) e_3 - (-1)^{n+3} (f_{n+2}+1) e_4 =$

$= (-1)^n [f_{n-1} + 1 + (f_n+1) e_2 + (f_{n+1}+1) e_3 + (f_{n+2}+1) e_4] =$

$= (-1)^n [F_{n-1} + 1 + e_2 + e_3 + e_4].\Box$

Proposition 2.2. *If* $h_n = pf_n + qf_{n+1} = 0$, *the following relation is true:*

$$H_n^2 = 3\frac{q^2}{f_n^2}\left[f_{2n+1} - f_{n+1}f_{n-2}f_{2n+2}\right]. \tag{2.2}$$

Proof. Since $h_n = 0$, it results that $t(H_n) = h_n = 0$, therefore $n(H_n) = H_n^2$. From $h_n = pf_n + qf_{n+1} = 0$, we have $p = -\frac{qf_{n+1}}{f_n}$ and we obtain:

$$p^2 + 2pq = \frac{q^2 f_{n+1}^2}{f_n^2} - 2q^2\frac{f_{n+1}}{f_n} = -\frac{q^2 f_{n+1}f_{n-2}}{f_n^2}$$

and

$$p^2 + q^2 = \frac{q^2 f_{n+1}^2}{f_n^2} + q^2 = q^2\frac{f_{n+1}^2 + f_n^2}{f_n^2} = q^2\frac{f_{2n+1}}{f_n^2},$$

since $f_{n+1}^2 + f_n^2 = f_{2n+1}$.

We obtain

$n(H_n) = 3[(p^2 + 2pq)f_{2n+2} + (p^2 + q^2)f_{2n+1}] =$

$= 3\frac{q^2}{f_n^2}[-f_{n+1}f_{n-2}f_{2n+2} + f_{2n+1}^2].\square$

3. Fibonacci-Narayana Quaternions

Theorem 3.1. *For the Fibonacci-Narayana quaternion U_n, we have*

$$a)\ \sum_{m=0}^{n} U_m = U_{n+3} - U_2,$$

$$b)\ \sum_{m=0}^{n} U_{3m} = U_{3n+1} - 1 - e_4.$$

Proof. a)

$$\sum_{m=0}^{n} U_m = \sum_{m=0}^{n} u_m + e_2\sum_{m=1}^{n+1} u_m + e_3\sum_{m=2}^{n+2} u_m + e_4\sum_{m=3}^{n+3} u_m =$$

using property 1 from Introduction, we have

$$= u_{n+3} - 1 + e_2(u_{n+4} - 1) + e_3(u_{n+5} - 2) + e_4(u_{n+6} - 3) =$$

$$= u_{n+3} - (1 + e_2 + 2e_3 + 3e_4) = u_{n+3} - u_2.$$

b)

$$\sum_{m=0}^{n} U_{3m} = \sum_{m=0}^{n} u_{3m} + e_2 \sum_{m=0}^{n} u_{3m+1} + e_3 \sum_{m=0}^{n} u_{3m+2} + e_4 \sum_{m=0}^{n} u_{3m+3} =$$

using properties 4, 2, 3, and again 4, we have

$$= u_{3n+1} - 1 + u_{3n+2}e_2 + u_{3n+3}e_3 + (u_{3n+4} - 1)e_4 = U_{3n+1} - 1 - e_4.$$

□

Let $\{U_n\}$ be a Fibonacci-Narayana sequens, and let $U_n = u_n \cdot 1 + u_{n+1}e_2 + u_{n+2}e_3 + u_{n+3}e_4$ be the nth Fibonacci-Narayana quaternion.

The function $f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n + \ldots$ is called *the generating function* for the sequence $\{a_0, a_1, a_2, \ldots\}$. In the paper [Ha; 12] was found a generating function for Fibonacci quaternions. In the following theorem, we established the generating function for Fibonacci-Narayana quaternions.

Theorem 3.2. *The generating function for the Fibonacci-Narayana quaternion U_n is*

$$G(t) = \frac{U_0 + (U_1 - U_0)t + (U_2 - U_1)t^2}{1 - t - t^3} = \frac{e_1 + e_2 + e_3 + (1 + e_3)t + (e_2 + e_3)t^2}{1 - t - t^3} \tag{3.1}$$

.

Proof. Assuming that the generating function of the quaternion Fibonacci-Narayana sequence $\{U_n\}$ has the form $G(t) = \sum_{n=0}^{\infty} U_n t^n$, it results that

$$U_0 + (U_1 - U_0)t + (U_2 - U_1)t^2 = \sum_{n=0}^{\infty} U_n t^n (1 - t - t^3),$$

or in equivalent form

$$\frac{U_0 + (U_1 - U_0)t + (U_2 - U_1)t^2}{1 - t - t^3} = \sum_{n=0}^{\infty} U_n t^n.$$

It is easy to see that the coefficients of t^0, t^1, t^2 are equal in the left and right side. The coefficients of t^n, $n \geq 3$ are equal with zero, since $0 = U_n - U_{n-1} - U_{n-3}$. The theorem is proved. □

Theorem 3.3. (The Binet-Cauchy formula for Fibonacci-Narayana numbers) *Let $u_n = u_{n-1} + u_{n-3}$ be the nth Fibonacci-Narayana number, then*

$$u_n = \frac{1}{(\alpha\text{-}\beta)(\beta\text{-}\gamma)(\gamma\text{-}\alpha)}\left[\alpha^{n+1}(\gamma\text{-}\beta) + \beta^{n+1}(\alpha\text{-}\gamma) + \gamma^{n+1}(\beta\text{-}\alpha)\right], \tag{3.2}$$

where α, β, γ are the solutions of the equation $t^3 - t^2 - 1 = 0$.

Proof. Supposing that $u_n = A\alpha^n + B\beta^n + C\gamma^n$, $A, B, C \in \mathbb{C}$ and using the recurence formula for the Fibonacci-Narayana numbers, it results that α, β, γ are the solutions of the equation $t^3 - t^2 - 1 = 0$. Since $u_0 = 0, u_1 = 1, u_2 = 1$,we obtain the following system

$$\begin{cases} A + B + C = 0 \\ A\alpha + B\beta + C\gamma = 1 \\ A\alpha^2 + B\beta^2 + C\gamma^2 = 1 \end{cases} \tag{3.3}$$

with the determinant $\Delta = (\alpha\text{-}\beta)(\beta\text{-}\gamma)(\gamma\text{-}\alpha) \neq 0$.

Using relations

$$\begin{cases} \alpha + \beta + \gamma = 1 \\ \alpha\beta + \alpha\gamma + \beta\gamma = 0 \\ \alpha\beta\gamma = 1 \end{cases},$$

by straightforward calculation, the solutions of this system (3) are

$A = \frac{\alpha(\gamma-\beta)}{(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)} = \frac{\alpha}{(\beta-\alpha)(\gamma-\alpha)}$

$B = \frac{\beta(\alpha-\gamma)}{(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)} = \frac{\beta}{(\alpha-\beta)(\gamma-\beta)}$

$C = \frac{\gamma(\beta-\alpha)}{(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)} = \frac{\gamma}{(\beta-\gamma)(\alpha-\gamma)}$, therefore the relation (3.2) is true. □

Theorem 3.4. (The Binet-Cauchy formula for the Fibonacci-Narayana quaternions) *Let $U_n = u_n \cdot 1 + u_{n+1}e_2 + u_{n+2}e_3 + u_{n+3}e_4$ be the nth Fibonacci-Narayana quaternion, then*

$$U_n = D\frac{\alpha^{n+1}}{(\beta\text{-}\alpha)(\gamma\text{-}\alpha)} + E\frac{\beta^{n+1}}{(\alpha\text{-}\beta)(\gamma\text{-}\beta)} + F\frac{\gamma^{n+1}}{(\beta\text{-}\gamma)(\alpha\text{-}\gamma)}, \tag{3.4}$$

where α, β, γ are the solutions of the equation $t^3 - t^2 - 1 = 0$ and
$D = 1 + \alpha e_1 + \alpha^2 e_2 + \alpha^3 e_3$,
$E = 1 + \beta e_1 + \beta^2 e_2 + \beta^3 e_3$,
$F = 1 + \gamma e_1 + \gamma^2 e_2 + \gamma^3 e_3$.

Proof. Using relation (3.2), we have that

$U_n = u_n \cdot 1 + u_{n+1}e_2 + u_{n+2}e_3 + u_{n+3}e_4 =$

$= \frac{1}{(\alpha\text{-}\beta)(\beta\text{-}\gamma)(\gamma\text{-}\alpha)}[(\alpha^{n+1}(\gamma\text{-}\beta) + \beta^{n+1}(\alpha\text{-}\gamma) + \gamma^{n+1}(\beta\text{-}\alpha)) \cdot 1 +$

$+(\alpha^{n+2}(\gamma-\beta) + \beta^{n+2}(\alpha-\gamma) + \gamma^{n+2}(\beta-\alpha))e_1 +$

$+(\alpha^{n+3}(\gamma-\beta) + \beta^{n+3}(\alpha-\gamma) + \gamma^{n+3}(\beta-\alpha))e_2 +$

$+(\alpha^{n+4}(\gamma-\beta) + \beta^{n+4}(\alpha-\gamma) + \gamma^{n+4}(\beta-\alpha))e_3] =$

$= \frac{1}{(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)}[\alpha^{n+1}(\gamma\text{-}\beta)(1+\alpha e_1+\alpha^2 e_2+\alpha^3 e_3)+$

$+\beta^{n+1}(\alpha\text{-}\gamma)(1+\beta e_1+\beta^2 e_2+\beta^3 e_3)+$

$+\gamma^{n+1}(\beta\text{-}\alpha)(1+\gamma e_1+\gamma^2 e_2+\gamma^3 e_3)].$

$\square$

Theorem 3.5. *Let $U_n = u_n \cdot 1 + u_{n+1}e_2 + u_{n+2}e_3 + u_{n+3}e_4$ be the nth Fibonacci-Narayana quaternion, therefore the following relations are true:*

1) $\sum\limits_{i=0}^{n} U_{pi} = U_{pn+p} - U_0.$

2) $\sum\limits_{i=0}^{n} C_n^i U_{2n-2i-1} = U_{3n-1}.$

3) $\sum\limits_{i=0}^{n} C_n^i U_{3n-2i-1} = U_{4n-1.}$

Proof. 1) With the above notations and using relation (3.4.), we have that:

$\sum\limits_{i=0}^{n} U_{pi} = U_0 + U_1 + ... + U_{pn} =$

$= \sum\limits_{i=0}^{n}\left(D\frac{\alpha^{n+1}}{(\beta\text{-}\alpha)(\gamma\text{-}\alpha)} + E\frac{\beta^{n+1}}{(\alpha\text{-}\beta)(\gamma\text{-}\beta)} + F\frac{\gamma^{n+1}}{(\beta\text{-}\gamma)(\alpha\text{-}\gamma)}\right) =$

$$= D\frac{1}{(\beta\text{-}\alpha)(\gamma\text{-}\alpha)}\frac{\alpha^{pn+p+1}-\alpha}{\alpha^p - 1} +$$

$$+ E\frac{1}{(\alpha\text{-}\beta)(\gamma\text{-}\beta)}\frac{\beta^{pn+p+1}-\beta}{\beta^p - 1} +$$

$$+ F\frac{1}{(\beta\text{-}\gamma)(\alpha\text{-}\gamma)}\frac{\gamma^{pn+p+1}-\gamma}{\gamma^p - 1} =$$

$= U_{pn+p} - U_0.$

2) Using the Newton's formula, it results that

$(t^2+1)^n = \mathtt{C}_n^0 (t^2)^n + \mathtt{C}_n^1 (t^2)^{n-1} + \mathtt{C}_n^2 (t^2)^{n-2} + ... + \mathtt{C}_n^n =$
$=\mathtt{C}_n^0 t^{2n} + \mathtt{C}_n^1 t^{2n-1} + \mathtt{C}_n^2 t^{2n-2} + ... + \mathtt{C}_n^n$. From here, we have that
$\sum\limits_{i=0}^{n} \mathtt{C}_n^i U_{2n-2i-1} = \mathtt{C}_n^0 U_{2n-1} + \mathtt{C}_n^1 U_{2n-3} + \mathtt{C}_n^2 U_{2n-5} + ... + \mathtt{C}_n^n U_{-1} =$
$=\mathtt{C}_n^0 \left(D\frac{\alpha^{2n}}{(\beta\text{-}\alpha)(\gamma\text{-}\alpha)} + E\frac{\beta^{2n}}{(\alpha\text{-}\beta)(\gamma\text{-}\beta)} + F\frac{\gamma^{2n}}{(\beta\text{-}\gamma)(\alpha\text{-}\gamma)} \right) +$
$+\mathtt{C}_n^1 \left(D\frac{\alpha^{2n-2}}{(\beta\text{-}\alpha)(\gamma\text{-}\alpha)} + E\frac{\beta^{2n-2}}{(\alpha\text{-}\beta)(\gamma\text{-}\beta)} + F\frac{\gamma^{2n-2}}{(\beta\text{-}\gamma)(\alpha\text{-}\gamma)} \right) + ... +$
$+\mathtt{C}_n^n \left(D\frac{1}{(\beta\text{-}\alpha)(\gamma\text{-}\alpha)} + E\frac{1}{(\alpha\text{-}\beta)(\gamma\text{-}\beta)} + F\frac{1}{(\beta\text{-}\gamma)(\alpha\text{-}\gamma)} \right) =$
$=D\frac{1}{(\beta\text{-}\alpha)(\gamma\text{-}\alpha)}(\mathtt{C}_n^0 \alpha^{2n} + \mathtt{C}_n^1 \alpha^{2n-2} + ... + \mathtt{C}_n^n 1) +$
$+E\frac{1}{(\alpha\text{-}\beta)(\gamma\text{-}\beta)}(\mathtt{C}_n^0 \beta^{2n} + \mathtt{C}_n^1 \beta^{2n-2} + ... + \mathtt{C}_n^n 1) +$
$+F\frac{1}{(\beta\text{-}\gamma)(\alpha\text{-}\gamma)}(\mathtt{C}_n^0 \gamma^{2n} + \mathtt{C}_n^1 \gamma^{2n-2} + ... \mathtt{C}_n^n 1) =$
$=D\frac{1}{(\beta\text{-}\alpha)(\gamma\text{-}\alpha)} (\alpha^2+1)^n + E\frac{1}{(\alpha\text{-}\beta)(\gamma\text{-}\beta)} (\beta^2+1)^n + F\frac{1}{(\beta\text{-}\gamma)(\alpha\text{-}\gamma)} (\gamma^2+1)^n =$
$=D\frac{1}{(\beta\text{-}\alpha)(\gamma\text{-}\alpha)} \alpha^{3n} + E\frac{1}{(\alpha\text{-}\beta)(\gamma\text{-}\beta)} \beta^{3n} + F\frac{1}{(\beta\text{-}\gamma)(\alpha\text{-}\gamma)} \gamma^{3n} = U_{3n-1}.$

3) Starting from relations $(t^3+t)^n = t^{4n}$, for $t \in \{\alpha, \beta, \gamma\}$, by straightforward calculations as in 2), we obtain the asked relation. $\square$

Conclusions. In this paper we investigated some new properties of generalised Fibonacci quaternions and Fibonacci-Narayana quaternions. Since Fibonacci-Narayana quaternions was not studied until now, we espect to find in the future more and surprising new properties.

References

[Cu; 76] I. Cucurezeanu, *Exercices in Arithmetics and Numbers Theory* (in Romanian), Ed. Tehnică, Bucharest, 1976.

[Di, St; 03] T. V. Didkivska, M. V. St'opochkina, *Properties of Fibonacci-Narayana numbers*, In the World of Mathematics, **9**(1)(2003), 29–36. [in Ukrainian]

[Dr, Gi, Gr, Wa; 03], A. Dress, R. Giegerich, S. Grűnewald, H. Wagner, *Fibonacci-Cayley Numbers and Repetition Patterns in Genomic DNA*, Ann. Comb. **7**(2003) 259–279.

[Ha; 12] S. Halici, *On Fibonacci Quaternions*, Adv. in Appl. Clifford Algebras, **22**(2)(2012), 321-327.

[Ho; 61] A. F. Horadam, *A Generalized Fibonacci Sequence*, Amer. Math. Monthly, **68**(1961), 455-459.

[Ho; 63] A. F. Horadam, *Complex Fibonacci Numbers and Fibonacci Quaternions*, Amer. Math. Monthly, **70**(1963), 289-291.

[Ko; 01] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, A Wiley-Interscience publication, U.S.A, 2001.

[Pa; 85] P. Singh, *The So-called Fibonacci Numbers in Ancient and Medieval India*, Historia Mathematica, **12**(1985), 229-244.

[Sa-Mu; 82] P. V. Satyanarayana Murthy, *Fibonacci-Cayley Numbers*, The Fibonacci Quarterly, **20**(1)(1982), 59-64.

[Sh; 06] V. Shpakivskyi, *On the one property of Fibonacci-Narayana numbers*, Mathematics, no. **11**(359)(2006), 12–14. [in Ukrainian]

[Sw; 73] M. N. S. Swamy, *On generalized Fibonacci Quaternions*, The Fibonacci Quaterly, **11(5)**(1973), 547-549.

[Yu; 61] A. P. Yushkevich, *History of Mathematics in the Middle Ages*, Gos. izd. fiz.-mat. lit., Moskow, 1961.

Cristina FLAUT
Faculty of Mathematics and Computer Science,
Ovidius University,
Bd. Mamaia 124, 900527, CONSTANTA,
ROMANIA
http://cristinaflaut.wikispaces.com/
http://www.univ-ovidius.ro/math/
e-mail:
cflaut@univ-ovidius.ro
cristina_flaut@yahoo.com

Vitalii SHPAKIVSKYI
Department of Complex Analysis and Potential Theory
Institute of Mathematics of the National Academy of Sciences of Ukraine,
3, Tereshchenkivs'ka st.
01601 Kiev-4
UKRAINE
http://www.imath.kiev.ua/~complex/
e-mail: shpakivskyi@mail.ru